ALL OF ZEROS OF RIEMANN'S ZETA-FUNCTION ARE ON $\sigma = 1/2$

NIANRONG FENG, YONGZHENG WANG, AND RUIXIN WU

Abstract. From Riemann's paper and nachlass, we obtained three conclusions about the zero of the Riemann zeta-function. First, we sort out the complete process of Riemann's proof of his hypothesis. In addition, a formula is revealed from a long-overlooked function,

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = 2\Re(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s)) \qquad \Re(s) = 1/2,$$

to compute the number of zeros of $\zeta(1/2+it)$. Furthermore, the evidence shows that Riemann had proved that all of zeros of Riemann's Zeta-function are on $\sigma = 1/2$.

1. Introduction

1.1. **Notions.** Let $s = \sigma + it$ be a complex number. Let the Riemann zeta-function be

$$\zeta(s) = \sum n^{-s} \qquad (\Re(s) > 1).$$

The strip $0 \le \Re(s) \le 1$ is called the critical strip and the vertical line $\Re(s) = 1/2$ is called the critical line. $\mathbf{D(T)}$ and $\mathbf{L(T)}$ are defined as

$$\mathbf{D}(T) = \{\sigma + it : 0 \le \sigma \le 1, 0 \le t \le T, \zeta(\sigma + iT) \ne 0\},$$
$$\mathbf{L}(T) = \{\sigma + it : \sigma = 1/2, 0 \le t \le T, \zeta(1/2 + iT) \ne 0\}.$$

Let $N_0(T)$ be the number of zeros of $\zeta(1/2+it)$ on $\mathbf{L}(T)$, and let $N(T)$ be the number of zeros of $\zeta(s)$ in $\mathbf{D}(T)$. The Riemann Hypothesis says that $N(T) = N_0(T)$.

1.2. **Introduction.** Riemann's work on the zero of the zeta function could be found in [1] and [2]. Though the author of [2] is C.L.Siegel, it mainly comes from Riemann's Nachlass(the Riemann Nachlass in this paper refers to [2]).

There are a large number of literature on [1] and [2] made a comprehensive and in-depth study[3, 4, 5, 10], but there are yet some content and formulas in [2] not being studied thoroughly. The most typical example is the following formula

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = 2\Re(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s)) \qquad \Re(s) = 1/2, \tag{1.1}$$

nobody has pointed out why Riemann gave it since 1932. In this paper, the evidence shows that Riemann had studied the zero of the zeta function from five aspects into three stages, which constitute a complete chain to prove the Riemann Hypothesis, and (1.1) is one of the key. The five parts appearing in [1] and [2] as follows:

(1) Analytic continuation for $\zeta(s)$.
(2) The distribution of zeros of $\zeta(s)$.
(3) Formula for calculating $N(T)$.

Date: January 24, 2017.
2010 *Mathematics Subject Classification.* 11M06; 11M26, 11H05.
Key words and phrases. Analytic continuation , zeta function , Complex conjugate , Non-trivial zeros.

(4) Formula for calculating the value of $\zeta(1/2+it)$.

(5) **Formula for calculating** $N_0(T)$**(which comes from** (1.1)**)**.

In order to fully understand Riemann's work, all the formulas about the zeta function in [1] and [2] are shown in the following chart.

Note: $\rightarrow$ on the graph represent the derivation. $\boxed{X}\rightarrow\boxed{Y}$: the formula Y is derived from the formula X. [1] $\sim$ [4] are well known formulas. All error terms are represented by $O(\frac{1}{t})$.

$$\Phi(u)=\int_{0\nwarrow 1}\frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x}-e^{-\pi i x}}dx$$

$$\Gamma(s)=\int_0^{\infty}e^{-x}x^{s-1}dx$$

Stage $\boxed{1}$

$$2\sin\pi s\Gamma(s)\zeta(s)=i\int_{-\infty}^{+\infty}\frac{x^{s-1}}{e^x-1}dx$$

[1]

$$\zeta(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x-1}dx$$

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s)$$

Functional equation

$$\Psi(x)=\sum_1^{\infty}e^{-n^2\pi x}$$

$$(A)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\frac{1}{s(s-1)}+\int_1^{\infty}\Psi(x)(x^{\frac{s}{2}-1}+x^{-\frac{s+1}{2}})dx$$

[2]

$\xi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)s(s-1)$, from $\xi(s)$ get N(T)$=\frac{\theta(T)+\arg\zeta(1/2+iT)}{\pi}+1$

$$(B)\xi(1/2+it)=4\int_1^{\infty}\frac{d(x^{3/2}\Psi(x)')}{dx}x^{-1/4}\cos(\frac{1}{2}t\ln x)dx$$

To prime

Riemann hypothesis(RH) is presented here.

Derivation the remainder for Z(t) and proof of RH

Stage $\boxed{2}$, Approximate proof of RH

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\sum_{n=1}^{M}n^{-s}+\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\sum_{n=1}^{M}n^{s-1}+O(\frac{1}{t}),M=\lfloor\sqrt{\frac{t}{2\pi}}\rfloor$$

[3]

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\varphi(s)+\varphi(1-s)+O(t^{-1}),\text{where } g(s)=\sum_{n=1}^{M}n^{-s},\varphi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})g(s)$$

Put s=1/2+it into the above, the two are obtained

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=2|\varphi(s)|\cos(\varphi(s))+O(\frac{1}{t})$$

$$Z(t)=2\sum_{n=1}^{M}\frac{\cos(\theta(t)-t\ln n)}{\sqrt{n}}+O(\frac{1}{t})$$

get $N_0(T)\approx\frac{\theta(T)+\arg\zeta(1/2+iT)}{\pi}+1$

It is a residual in $O(\frac{1}{t})$

$$\int_{0\nwarrow 1}\frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x}-e^{-\pi i x}}dx=\frac{1}{1-e^{-2\pi i u}}-\frac{e^{\pi i u^2}}{e^{\pi i u}-e^{-\pi i u}}$$

$$\frac{\cos(2\pi u^2+3\pi/8)}{\cos(2\pi u)}$$

Stage $\boxed{3}$

2

multiplying each side by $\int_0^{\bar{\varepsilon}\infty} u^{-s}du$

$$\boxed{\int_0^{\bar{\varepsilon}\infty} u^{-s}du \int_{0\nwarrow 1} \frac{e^{-\pi i x^2+2\pi i u x}}{e^{\pi i x}-e^{-\pi i x}}dx = \int_0^{\bar{\varepsilon}\infty} u^{-s}\frac{1}{1-e^{-2\pi i u}}du - \int_0^{\bar{\varepsilon}\infty} u^{-s}\frac{e^{\pi i u^2}}{e^{\pi i u}-e^{-\pi i u}}du}$$

$$\boxed{\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx + \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\int_{0\searrow 1}\frac{e^{-\pi i x^2}x^{s-1}}{e^{\pi i x}-e^{-\pi i x}}dx} \text{[4]}$$

$$\boxed{g(s) = \int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx, \quad \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})g(s) + \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})g(1-s)}$$

$$\boxed{\text{Set} \quad \varphi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})g(s), \text{ then } \quad \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(s) = \varphi(s)+\varphi(1-s)}$$

Put s=1/2+it into above, according to complex conjugate

$$\boxed{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = 2\Re[\varphi(s)]. \ \underline{\text{This conclusion is given by Riemann !}}}$$

By De Moivre's theorem

$$\boxed{\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = 2|\varphi(1/2+it)|\cos[\varphi(s)]}$$

$$\boxed{e^{i\theta(t)}\zeta(1/2+it) = 2|g(1/2+it)|\cos[\theta(t)+\arg g(1/2+it)]}$$

By $\cos(x) = 0$

$$\boxed{\text{when} \quad 0 < t \le T, \text{ we obtain} \quad N_0(T) = \frac{\theta(T)+\arg\zeta(1/2+iT)}{\pi}+1, \text{namely } N(T) = N_0(T)}$$

In this paper, investigating the geometric meaning of (1.1) and going through the proof chain, we got the formula $N_0(T)$ and $N(T) = N_0(T)$.

2. Solved problems in Riemann's paper

2.1. **Analytic continuation for $\zeta(s)$.** Euler's Gamma function is

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx. \tag{2.1}$$

According to the formula, the following formula is finally obtained

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}dx. \tag{2.2}$$

By

$$\int_c \frac{(-x)^{s-1}}{e^x-1}dx = (e^{-\pi s i}-e^{\pi s i})\int_0^\infty \frac{x^{s-1}}{e^x-1}dx$$

and (2.2), one obtains

$$2 \sin \pi s \Gamma(s) \zeta(s) = i \int_c \frac{(-x)^{s-1}}{e^x - 1} dx. \tag{2.3}$$

Riemann provided[1] that, **in the many-valued function** $(-x)^{s-1} = e^{(s-1)\log(-x)}$**,** $\log(-x)$ **is determined so as to be real when** x **is negative. This equation now shows that this function** $\zeta(s)$ **is one-valued, and** $s = 1$ **is its a simple pole**.

2.2. **The distribution of zeros of** $\zeta(s)$**.** It is known that[10]

$$\zeta(-n) = -\frac{(-1)^n}{n+1} B_{n+1},$$

where $n = 1, 2, 3, \cdots$, B_n is the Bernoulli numbers and $B_{2n+1} = 0$. **This is clearly** $\zeta(s)$ **is zero if s is equal to a negative even integer**.

By using (2.1) and 2.3, the functional equation of the zeta function is obtained

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{2.4}$$

It known that the only poles of $\Gamma(s)$ are simple and situated at $s = 0, -1, -2, \cdots$. $\zeta(s)$ has simple zeros at $s = -2, -4, \cdots$, which are cancelled by the poles of $\Gamma(\frac{s}{2})$ other than the pole of s=0. Therefore, $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$ has no zeros or poles in the left side of the complex plane.

In view of this (2.4), if there exists s_0 such that $\zeta(s_0) = 0$, where $s_0 \neq 0, 1$ and $\Re(s_0) \neq \frac{1}{2}$, then

$$\zeta(1 - s_0) = \overline{\zeta(1 - \overline{s_0})} = 0.$$

This shows that $\zeta(s)$ is symmetric around the vertical line $\Re(s) = \frac{1}{2}$. Namely, in $D(T) \backslash L(T)$, the distribution of zeros of $\zeta(s)$ satisfy

$$N(T) - N_0(T) = 2M,$$

where $M \geq 0$ is an integer.

With this, we get the zeros of $\underline{\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)}$ are all lie in the critical strip except $s = 0, 1$.

2.3. **Formula for calculating** $N(T)$**.** It is the key to obtain the formula for calculating $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$. To this end, according to the formula (2.1) and the functional equation for $\vartheta(x)$,

$$x^{1/2} \vartheta(x) = \vartheta(\frac{1}{x}), \quad \vartheta(x) = \sum e^{-n^2 \pi x}$$

the following formula is derived

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \vartheta(x) (x^{\frac{s}{2}-1} + x^{-\frac{s+1}{2}}) dx. \tag{2.5}$$

Multiply both sides of the above by $s(s-1)/2$, then

$$\frac{s}{2}(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \frac{1}{2} + \frac{s}{2}(s-1) \int_1^\infty \Psi(x) (x^{\frac{s}{2}-1} + x^{-\frac{s+1}{2}}) dx. \tag{2.6}$$

We now set

$$\xi(s) = \frac{s}{2}(s-1) \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s).$$

It shows that $\xi(s)$ is an entire function without poles, and $\xi(s)$ has the same non-trivial zeros as $\zeta(s)$ in the critical strip. This formula (2.6) is not the second proof of the functional equation, it has to solve the two problem:

2.3.1. *The distribution of zeros of $\zeta(s)$ in $D(T)$.* By (2.6), Riemann provided the approximately value for $N(T)$, and stated below:

The number of roots of $\xi(t) = 0$, whose real parts lie between 0 and T is approximately

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi};$$

because the integral $\int d\log\xi(t)$,taken in a positive sense around the region consisting of th values of t whose imaginary parts lie between $\frac{1}{2}i$ and $-\frac{1}{2}i$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $\frac{1}{T}$)equal to $(T\log\frac{T}{2\pi} - T)i$, this integral however is equal to the number of roots of $\xi(t) = 0$ lying within in this region,multiplied by $2\pi i$.

He clearly pointed out that the number of zeros of $\xi(s)$ are obtained by the argument principle.

Theorem 2.1. *Within $G(T) = \{\sigma, t \in R : -1 \le \sigma \le 2, 0 \le t \le T, \zeta(\sigma + iT) \ne 0\}(D(T) \in G(T))$, the number of zeros of $\xi(s)$*

$$N(T) = \frac{\theta(T) + \arg\zeta(1/2 + T)}{\pi} + 1,$$

where

$$\begin{aligned}\theta(T) &= \arg[\pi^{-\frac{(1/2+iT)}{2}}\Gamma(\frac{1/2 + iT}{2})] \\ &= \frac{T}{2}\log\frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + O(T^{-1}).\end{aligned}$$

Proof. Let R be the positively oriented rectangular contour with $G(T)$. By the argument principle we have,

$$N(T) = \frac{1}{2\pi}\triangle_R \arg\xi(s). \tag{2.7}$$

$\triangle_R \arg\xi(s)$ counts the changes in the argument of $\xi(s)$ along the contour R(see [7]). We divide R into three sub-contours. Let $L1$ be the horizontal line from -1 to 2. Let $L2$ be the sub-contour from 2 to $2 + iT$ and then to $1/2 + iT$. Finally, let $L3$ be the sub-contour from $1/2 + iT$ to $-1 + iT$ and then -1. Accordingly,

$$\triangle_R \arg\xi(s) = \triangle_{L_1}\arg\xi(s) + \triangle_{L_2}\arg\xi(s) + \triangle_{L_3}\arg\xi(s) \tag{2.8}$$

By $\xi(s) = \xi(1 - s)$ we have $\xi(s) = \overline{\xi(1 - \overline{s})}$. It shows that

$$\triangle_{L_2}\arg\xi(s) = \triangle_{L_3}\arg\xi(s) \tag{2.9}$$

Along the L_1 since $\xi(s)$ is a real function, thus

$$\triangle_{L_1}\arg\xi(s) = 0. \tag{2.10}$$

According to (2.7)-(2.10), we have

$$N(T) = \frac{1}{\pi}\triangle_{L_2} \arg \xi(s). \tag{2.11}$$

It is known that

$$\triangle_{L_2} \arg \xi(s) = \triangle_{L_2} \arg [\frac{s}{2}(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)].$$

We consider $\triangle_{L_2} \arg [\frac{s}{2}(s-1)]$,

$$\begin{aligned}\triangle_{L_2} \arg [\frac{s}{2}(s-1)] &= \arg(1/2+iT) - \arg 2 + \arg(-1/2+iT) - \arg 2 \qquad (\arg 2 = 0)\\ &= \pi.\end{aligned}$$

Next we consider $\triangle_{L_2} \arg [\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)]$,

$$\begin{aligned}\triangle_{L_2} \arg [\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)] &= \triangle_{L_2} \arg [\pi^{-s/2}\Gamma(\frac{s}{2})] + \triangle_{L_2} \arg \zeta(s) - \arg 2\\ &= \arg [\pi^{-\frac{1/2+iT}{2}}\Gamma(\frac{1/2+iT}{2})] + \arg \zeta(1/2+iT)\\ &= \theta(T) + \arg \zeta(1/2+iT).\end{aligned}$$

By the two above and (2.11), we obtain

$$N(T) = \frac{1}{\pi}\triangle_{L_2} \arg \xi(s) = \frac{\theta(T) + \zeta(1/2+iT)}{\pi} + 1.$$

□

Corollary 2.1. If $t \in L(T)$ such that $\zeta(1/2+it) \neq 0$, then $\theta(t) + \zeta(1/2+it)$ is an integer multiple of π.

2.3.2. *The distribution of zeros of $\zeta(s)$ on $L(T)$.* Now we set $s = 1/2+it$ and put it into (2.6), so that

$$\xi(t) = \int_1^\infty \frac{d(x^{\frac{3}{2}}\Psi'(x))}{dx} x^{-\frac{1}{4}} \cos(\frac{1}{2}t\log x)dx.$$

Riemann didn't give a precise value for the above, and proposed the Riemann Hypothesis. However, this factor $\cos(\frac{1}{2}t\log x)$ revealed two important information: On the one hand, the zeros of $\xi(t)$ is related to the periodicity of $\cos x$, but on the other hand, $x^{\frac{s}{2}-1}$ and $x^{-\frac{1+s}{2}}$ in (2.6) are a pair of complex conjugate only if $s = 1/2+it$. This means that, the zeros of $\xi(s)$ can be calculated by $\cos x$ as long as $\xi(s)$ is converted into

$$\xi(s) = F(s) + F(1-s).$$

The content in Riemann's Nachlass is exactly the same as the speculation.

3.1. **Formula for the value of $\zeta(1/2+it)$ and proof of $N(T) \approx N_0(T)$.** The reason why we believe that Riemann's Nachlass is the continuation for his the paper, he started with (2.3) and got

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\sum_{n=1}^{M}\frac{1}{n^s} + \pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})\sum_{n=1}^{M}\frac{1}{n^{1-s}} + R(s), \tag{3.1}$$

where $R(s)$ be an error term, $M = \lfloor\sqrt{\frac{t}{2\pi}}\rfloor$. Putting $s = 1/2+it$ into the above, we have

$$e^{i\theta(t)}\zeta(1/2+it) = e^{i\theta(t)}\sum_{n=1}^{M}\frac{1}{n^{1/2+it}} + e^{-i\theta(t)}\sum_{n=1}^{M}\frac{1}{n^{1/2-it}} + O(t^{-1/4}). \tag{3.2}$$

By the conjugate properties

$$e^{i\theta(t)}\zeta(1/2+it) = 2\sum_{n=1}^{M}\frac{1}{\sqrt{n}}\cos(\theta(t) - t\ln n) + O(t^{-1/4}). \tag{3.3}$$

The above is a well-known formula for calculating the value of $\zeta(1/2+it)$. By Jacobi's function

$$\Phi(\tau, u) = \int \frac{e^{\pi i\tau x^2 + 2\pi iux}}{e^{2\pi ix} - 1}dx,$$

one obtains

$$\int_{0\nwarrow 1}\frac{e^{-\pi ix^2+2\pi iux}}{e^{\pi ix} - e^{-\pi ix}}dx = \frac{1}{1-e^{-2\pi iu}} - \frac{e^{\pi iu^2}}{e^{\pi iu} - e^{-\pi iu}}, \tag{3.4}$$

and then

$$\int_{0\nwarrow 1}\frac{e^{\pi i[x^2-2(u+1/2-x)^2+1/8]}}{e^{2\pi ix} - 1}dx = \frac{\cos(2\pi u^2 + 3\pi/8)}{\cos(2\pi u)}.$$

The above is a key factor of $O(t^{-1/4})$.

Riemann[2] in a letter to the Weierstrass stressed that he had found a way to prove the non-trivial zeros of $\zeta(s)$ are almost on the critical line. This conclusion can be obtained from (3.1).

Theorem 3.1. *The non-trivial zeros of $\zeta(s)$ in $D(T)$ are almost on the $L(T)$.*

Proof. By (3.1), we set

$$\mu(s) = \sum_{n=1}^{M}\frac{1}{n^s},$$

and putting it into (3.2), we have

$$e^{i\theta(t)}\zeta(1/2+it) = e^{i\theta(t)}\mu(1/2+it) + e^{-i\theta(t)}\mu(1/2-it) + O(t^{-1/4}) \tag{3.5}$$

By the conjugate properties, we obtain

$$\begin{aligned} e^{i\theta(t)}\zeta(1/2+it) &= 2\Re(e^{i\theta(t)}\mu(1/2+it)) + O(t^{-1/4}) \\ &= 2|f(1/2+it)|\cos[\theta(t) + \arg\mu(1/2+it)] + O(t^{-1/4}) \end{aligned}$$

$$(3.6)\qquad e^{i\theta(t)}\zeta(1/2+it) \approx 2|\mu(1/2+it)|\cos[\theta(t)+\arg\mu(1/2+it)].$$

On the $L(T)$, the zeros of $\cos[\theta(t)+\arg\mu(1/2+it)]$ correspond to the zeros of $\zeta(1/2+it)$, which is equal to

$$\frac{\theta(T)+\arg\mu(1/2+iT)}{\pi}+\frac{1}{2}.$$

The approximate value of the above is exactly the same as that given by Riemann. □

Since $O(t^{-1/4})$ in (3.5), Riemann had to use the word "almost". That's natural that the next work is to eliminate it.

3.2. **Function and geometric meaning of the integral formula.** Known that

$$\int_{0\nwarrow 1}\frac{e^{-\pi ix^2+2\pi iux}}{e^{\pi ix}-e^{-\pi ix}}dx=\frac{1}{1-e^{-2\pi iu}}-\frac{e^{\pi iu^2}}{e^{\pi iu}-e^{-\pi iu}}.$$

Multiply both sides of the above by $\int u^{-s}du$, and integrate along the ray from 0 to $e^{\frac{\pi i}{4}}\infty$, u^{-s} is defined on the slit plane(excluding 0 and $-\infty$). Finally, he obtained(see [2])

$$(3.7)\quad \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\int_{0\swarrow 1}\frac{x^{-s}e^{\pi ix^2}}{e^{\pi ix}-e^{-\pi ix}}dx+\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\int_{0\searrow 1}\frac{e^{-\pi ix^2}x^{s-1}}{e^{\pi ix}-e^{-\pi ix}}dx$$

This above is known as the Riemann-Siegel integral formula. By using the residue theorem, we have

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})[\sum_{n=1}^{M}\frac{1}{n^s}+R_1(s)]+\pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})[\sum_{n=1}^{M}\frac{1}{n^{1-s}}+R_2(s)]$$

Comparing (3.1) and the above, we see that the latter is not independent of the error term. This shows that Riemann's purpose is to eliminate $R(s)$.

Now we set

$$f(s)=\int_{0\swarrow 1}\frac{x^{-s}e^{\pi ix^2}}{e^{\pi ix}-e^{-\pi ix}}dx,$$

and take it into (3.7), then

$$(3.8)\qquad \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s)+\pi^{\frac{s-1}{2}}\Gamma(\frac{1-s}{2})f(1-s).$$

By the above, Riemann also derived an important functional formula. His derivation is as follows:

Putting $s=1/2+it$ **into** (3.8),

$$(3.9)\qquad e^{i\theta(t)}\zeta(1/2+it)=e^{i\theta(t)}f(1/2+it)+e^{-i\theta(t)}f(1/2-it)$$

From the above formula, finally

$$(3.10)\qquad e^{i\theta(t)}\zeta(1/2+it)=2\Re[\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s)],\qquad s=1/2+it$$

This above still sleeps in Riemann's Nachlass. Figure 1 is a fragment of [2]. It is seen that C.L.Siegel was aware of the importance of (3.10), and asserted that $\varphi(s)$ is the key to study the zero of $\zeta(s)$. To understand the role of (3.8), one must know the geometric meaning of (3.10).

$$(56)\qquad \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)=\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}\,dx + \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\int_{0\searrow 1}\frac{x^{s-1}e^{-\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}\,dx.$$

$$(57)\qquad \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Setzt man noch

$$(58)\qquad f(s)=\int_{0\swarrow 1}\frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}\,dx$$

$$(59)\qquad \varphi(s)=2\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)f(s),$$

so gilt nach (56) und (57)

$$(60)\qquad \underline{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\Re\left(\varphi(s)\right)}\qquad \left(\sigma=\frac{1}{2}\right):$$

damit ist die Untersuchung von $\zeta(s)$ auf der kritischen Geraden zurückgeführt auf die Untersuchung des reellen Teils von $\varphi(s)$.

FIGURE 1. The fragments from [2]

Let

$$\varphi(s)=\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s),$$

by (3.8), we have

$$(3.11)\qquad \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\varphi(s)+\varphi(1-s).$$

By the functional equation, $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ is a real when $s=1/2+it$. From (3.8), we have

$$(3.12)\qquad e^{i\theta(t)}\zeta(1/2+it)=e^{i\theta(t)}f(1/2+it)+e^{-i\theta(t)}f(1/2-it),$$

and

$$\varphi(1/2+it)=\overline{\varphi(1/2-it)},$$

where $\varphi(1/2-it)=e^{-i\theta(t)}\overline{f(1/2+it)}$.

The geometric meaning of the above is shown in Figure 2. Between $\arg[e^{i\theta(t)}\zeta(1/2+it)]$ and $\arg[e^{i\theta(t)}f(1/2+it)]$ must meet

$$-\frac{\pi}{2}\le \arg[e^{i\theta(t)}f(1/2+it)]-\arg[e^{i\theta(t)}\zeta(1/2+it)]\le\frac{\pi}{2},$$

namely,

$$(3.13)\qquad \theta(t)+\arg\zeta(1/2+it)-\frac{\pi}{2}\le\theta(t)+\arg f(1/2+it)\le\theta(t)+\arg\zeta(1/2+it)+\frac{\pi}{2}.$$

The above shall become equality if and only if $\zeta(1/2+it)=0$.

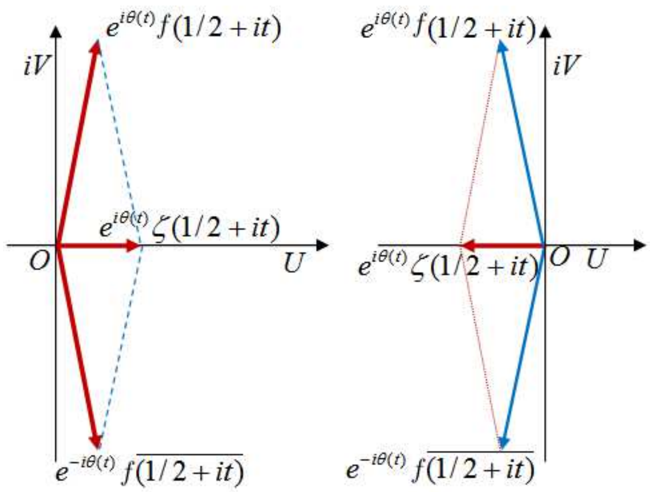

FIGURE 2. The geometric of (3.12), when $s = 1/2 + it$

4. The derivation of $N_0(T)$ and proof of $N(T) = N_0(T)$

According to De Moivre's theorem, from (3.10) we get

$$e^{i\theta(t)}\zeta(1/2+it) = 2r(t)\cos[\theta(t) + \arg f(1/2+it)], \tag{4.1}$$

where $r(t) = |\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})f(s)|$,

$$\begin{aligned}\theta(t) &= \arg\,[\pi^{-\frac{(1/2+it)}{2}}\Gamma(\frac{1/2+it}{2})] \\ &= \frac{t}{2}\log\frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + O(t^{-1}).\end{aligned}$$

Theorem 4.1. *On the $L(T)$, the number of zeros of $\zeta(1/2+it)$*

$$N_0(T) = \frac{\theta(T) + \arg\zeta(1/2+iT)}{\pi} + 1.$$

Proof. Set

$$\omega(t) = \theta(t) + \arg f(1/2+it).$$

For the convenience of derivation, we use the interval $[0,T]$ instead of $L(T)$. In view of (4.1), the zeros of $r(t)$ and $\cos[\omega(t)]$ are all the zeros of $\zeta(1/2+it)$. Let $N_r(T)$ be the number of zeros of $r(t) = 0$ and $\cos[\omega(t)] \neq 0$, and let $N_c(T)$ be the number of zeros of $\cos[\omega(t)] = 0$. Thus we have

$$N_0(T) = N_c(T) + N_r(T).$$

$\cos[\omega(t)]$ is a composite function, we must obtain the range of values of $\omega(t)$ on $[0,T]$.

When $s = 1/2 + 0i$, $\pi^{-s/2}\Gamma(\frac{s}{2}) > 0$ is a real, and

$$\arg\pi^{-s/2}\Gamma(\frac{s}{2}) = \theta(0) = 0.$$

We know that $\zeta(1/2+0i) \approx -1.46$ is a real number, $\arg\zeta(1/2+0i) = -\pi$. Thus

$$\theta(0) + \arg\zeta(1/2+0i) = -\pi. \tag{4.2}$$

Putting $t = 0$ into (3.13), we get

$$\theta(0) + \zeta(1/2 + 0i) - \frac{\pi}{2} < \omega(0) < \theta(0) + \zeta(1/2 + 0i) + \frac{\pi}{2}.$$

By (4.2), we have

$$-\frac{3\pi}{2} < \omega(0) < -\frac{\pi}{2}.$$

When $s = 1/2 + iT$, Taking $t = T$ into (3.13), we get

$$\theta(T) + \arg\zeta(1/2 + iT) - \frac{\pi}{2} < \omega(T) < \theta(T) + \arg\zeta(1/2 + iT) + \frac{\pi}{2}. \tag{4.3}$$

In summary, the range of values of $\omega(t)$ on $[0, T]$ is

$$(-\frac{3\pi}{2}, \theta(T) + \arg\zeta(1/2 + iT) + \frac{\pi}{2}).$$

By the intermediate value theorem, there exists $t_m \in [0, T]$ such that $\omega(t_m) = 0$, and for all $t > t_m$, $\omega(t) > 0$. it follows that

$$[0, T] = [0, t_m] \cup (t_m, T],$$

and

$$(-\frac{3\pi}{2}, \theta(T) + \arg\zeta(1/2 + iT) + \frac{\pi}{2}) = (-\frac{3\pi}{2}, 0] \cup (0, \theta(T) + \arg\zeta(1/2 + iT) + \frac{\pi}{2}).$$

Let $N_m(T)$ be the number of zeros of $\cos[\omega(t)]$ on $[0, t_m]$, and let $\Bbbk(T)$ be the number of zeros of $\cos[\omega(t)]$ on $(t_m, T]$. Accordingly, the number of zeros of $\cos[\omega(t)]$ on $[0, T]$

$$N_c(T) = N_m(T) + \Bbbk(T).$$

For $-\frac{\pi}{2} \in (-\frac{3\pi}{2}, 0]$, there is at least $t_x \in [0, t_m]$ such that

$$\omega(t_x) = -\frac{\pi}{2}, \quad \cos[\omega(t_x)] = 0.$$

Clearly

$$N_m(T) \geq 1. \tag{4.4}$$

For all $t \in [t_m, T]$, these real numbers such as $\omega(t) = n \cdot \pi - \frac{\pi}{2}$ are all the zero of $\cos[\omega(t)]$. Obviously, the number of zeros of $\cos[\omega(t)]$ on $(t_m, T]$

$$\Bbbk(T) = \lfloor \frac{\omega(T)}{\pi} + \frac{1}{2} \rfloor. \tag{4.5}$$

By (4.3), we have

$$\theta(T) + \arg\zeta(1/2 + iT) < \omega(T) + \frac{\pi}{2} < \theta(T) + \arg\zeta(1/2 + iT) + \pi,$$

namely,

$$\frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} < \frac{\omega(T)}{\pi} + \frac{1}{2} < \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} + 1.$$

By the above and Corollary 2.1, we get

$$\begin{aligned}\Bbbk(T) &= \lfloor \frac{\omega(T)}{\pi} + \frac{1}{2} \rfloor \\ &= \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi}.\end{aligned}$$

It follows that

$$\begin{aligned}N_c(T) &= N_m(T) + \Bbbk(T) \\ &= N_m(T) + \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi},\end{aligned}$$

also

$$\begin{aligned}N_0(T) &= N_c(T) + N_r(T) \\ &= \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} + N_m(T) + N_r(T).\end{aligned}$$

For

$$N_0(T) \leq N(T),$$

by expression of $N_0(T)$ and $N(T)$, we have

$$N_m(T) + N_r(T) + \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} \leq \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} + 1,$$

namly

$$N_m(T) + N_r(T) \leq 1.$$

By (4.4), we have

$$N_m(T) + N_r(T) \geq 1.$$

Hence, if and only if $N_m(T) = 1, N_r(T) = 0$, the above two inequalities can be established at the same time. Comprehensive above proof, we get

$$N_0(T) = \frac{\theta(T) + \arg\zeta(1/2 + iT)}{\pi} + 1$$

□

5. CONCLUSIONS

According to the theorems (2.1) and (4.1), we now prove that

$$\underline{N(T) = N_0(T)}.$$

By the arbitrariness of T, when $T \to \infty$ all non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = 1/2$.

6. Acknowledgments

I would like to express acknowledgement here to the Delaware State University Mr. Yi Ling for his help. Thanks to all those friends and colleagues who concern and support us.

References

[1] Bernhard Riemann. Über die Anzahl der Primzahlen unter einer gegebenen Grosse. Monatsberichte der Berliner Akademie, 1859.

[2] C.L. Siegel, Über Riemanns Nachlass zur analytischen Zahlentheorie,Quellen und Studien zur Geschichte der Mathematik, Gesammelte Abhandlungen, Bd. I, Springer-Verlag, Berlin, 1966, 275-310.

[3] N. Levinson, More than One Third of Zeros of Riemann's Zeta-Function are on G = l/2,ADVANCES IN MATHEMATICS 13, 383-436 (1974)

[4] J. B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, J.Reine Angew. Math. 399 (1989), 1-26.

[5] Harold M. Edwards. Riemann's Zeta Function. Dover, 1974.

[6] J.B. Conrey, D.W. Farmer, Mean values of L-functions and symmetry, Int. Math. Res. Not. 17 (2000) 883- 908.

[7] Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller. The Riemann Hypothesis. July 12, 2006

[8] A. Selberg. On the remainder in the formula for N(T), the number of zeros of $\zeta(s)$ in the strip $0 < t < T$ .In Collected papers of A.Selberg, volume 1, pages 179-203. 1944. The year of the original publication is 1944.

[9] Juan Arias de Reyna and Jan van de Lune,A first encounter with the Riemann Hypothesis and its numerical verificationLa Gaceta de la RSME, Vol. 00 (0000), Num. 0, Pags. 1-24

[10] H. Vic Dannon, Riemann's Zeta Paper: The Product Formula Error, Gauge Institute Journal May 2006.

School of Mathematics & Computer Science,Anhui Normal University, Wuhu, Anhui PR China

E-mail address: fengnr@mail.ahnu.edu.cn

School of Mathematics & Computer Science,Anhui Normal University, Wuhu, Anhui PR China

E-mail address: fnr348@126.com

School of Mathematics & Computer Science,Anhui Normal University, Wuhu, Anhui PR China

E-mail address: ruixinwu@mail.ahnu.edu.cn